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ASYMPTOTIC ANALYSIS OF STATIONARY PROPAGATION OF THE FRONT OF A TWO-STAGE EXOTHERMIC REACTION IN A GAS

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We construct an approximate solution of the problem concerning the propagation of a planar front of a two-stage exothermic sequential chemical reaction in a gas, by the method of matched asymptotic expansions. As the parameter in the expansion we use the ratio of the adiabatic combustion temperature to the sum of the activation temperatures of both reactions. Depending on the values of the characteristic parameters of the problem, we consider several solutions, each with a different asymptotic behavior, corresponding to the various flame front propagation modes. The analytical results obtained are compared with numerical data available in the literature.

1. Formulation of the problem. The stationary propagation of a planar front of a two-stage sequential exothermic reaction $A_1 \to A_2 \to A_3$ in a gas can, subject to a number of simplifying assumptions, be described by the following equations and boundary conditions:

$$\frac{\lambda}{c}\frac{d^2T}{dx^2} - m\frac{dT}{dx} + \frac{Q_1}{c}a_1\rho k_1 \exp\frac{-E_1}{RT} + \frac{Q_2}{c}a_2\rho k_2 \exp\frac{-E_2}{RT} = 0 \quad (1.1)$$

$$\rho D\frac{d^2a_1}{dx^2} - m\frac{da_1}{dx} - a_1\rho k_1 \exp\frac{-E_1}{RT} = 0 \quad (1.2)$$

$$\rho D\frac{d^2a_2}{dx^2} - m\frac{da_2}{dx} + a_1\rho k_1 \exp\frac{-E_1}{RT} - a_2\rho k_2 \exp\frac{-E_2}{RT} = 0 \quad (1.3)$$

$$x = -\infty, \quad T = T_-, \quad a_1 = 1, \quad a_2 = 0 \tag{1.4}$$

$$x = \infty, \quad a_1 = a_2 = 0, \quad dT / dx = 0 \tag{1.5}$$

Here x is the spatial coordinate, a_1 and a_2 are the mass fractions of the substance A_1 and A_2, T is the temperature, ρ is the density, m is the combustion mass rate, c is the heat capacity, λ is the thermal conductivity, R is the gas constant, Q_1 and Q_2 are thermal reaction effects, k_1 and k_2 are the factors premultiplying the exponential terms, E_1 and E_2 are the activation energies, and D is the diffusion coefficient of the substances A_1 and A_2. We assume that the density and all the thermophysical characteristics of the medium maintain constant values.

Solution of the problem (1.1) – (1.5) consists in determining the functions $a_1(x)$, $a_2(x)$ and $T(x)$, and of the eigenvalue of m. For a solution to exist it is sufficient that the constant k_1 be set equal to zero over a small interval close to T_- [1].

The problem (1.1) – (1.5) has the first integral

$$\frac{\lambda}{mc}\frac{dT}{dx} = T - T_+ + \frac{Q_1 + Q_2}{c}\left[a_1 - \frac{D\rho}{m}\frac{da_1}{dx}\right] + \frac{Q_2}{c}\left[a_2 - \frac{D\rho}{m}\frac{da_2}{dx}\right], \quad T_+ = T_- + c^{-1}(Q_1 + Q_2) \tag{1.6}$$

Taking (1.6) into account, we can represent the problem (1.1) – (1.5) in the following form:

$$\frac{dr}{d\tau} = \frac{L(r - H)}{\tau - \sigma_Q H - (1 - \sigma_Q) G} \tag{1.7}$$

$$\frac{dq}{d\tau} = \frac{L(q - G)}{\tau - \sigma_Q H - (1 - \sigma_Q) G} \tag{1.8}$$

$$\mu\frac{dH}{d\tau} = \frac{\sigma_k(1 - r)}{\tau - \sigma_Q H - (1 - \sigma_Q) G}\exp\frac{-\beta\sigma_E(1 + \sigma)}{\tau + \sigma} \tag{1.9}$$

$$\mu\frac{dG}{d\tau} = \frac{(1 - \sigma_k)(r - q)}{\tau - \sigma_Q H - (1 - \sigma_Q) G}\exp\frac{-\beta(1 - \sigma_E)(1 + \sigma)}{\tau + \sigma} \tag{1.10}$$

$$\tau = 0, \qquad r = q = G = H = 0 \tag{1.11}$$

$$\tau = 1, \qquad r = q = G = H = 1 \tag{1.12}$$

The variable τ, the unknown functions r, q, H, G, the eigenvalue μ and the dimensionless constants L, σ_Q, σ_k, σ_E, β and σ are determined by the formulas

$$\tau = \frac{T - T_-}{T_+ - T_-}, \quad r = 1 - a_1, \quad q = 1 - a_1 - a_2, \quad L = \frac{\lambda}{\rho D c} \tag{1.13}$$

$$G(\tau) = q - \frac{\rho D}{m}\frac{dq}{dx}, \quad H(\tau) = r - \frac{\rho D}{m}\frac{dr}{dx}, \quad \sigma_k = \frac{k_1}{k_1 + k_2},$$

$$\sigma_E = \frac{E_1}{E_1 + E_2}, \quad \sigma_Q = \frac{Q_1}{Q_1 + Q_2}, \quad \beta = \frac{E_1 + E_2}{RT_+}$$

$$\sigma = \frac{T_-}{T_+ - T_-}, \qquad \mu = \frac{m^2 c}{\lambda\rho(k_1 + k_2)}$$

From the conditions of nonnegativity of the concentration, the conversion sequence of the reagents, and from the condition of thermal gradient nonnegativity we have the following inequalities:

$$\tau - \sigma_Q H - (1 - \sigma_Q)G \geqslant 0, \quad r \geqslant H \geqslant 0, \; q \geqslant G \geqslant 0 \qquad (1.14)$$
$$1 \geqslant r \geqslant q \geqslant 0$$

To construct approximate analytical solutions of the problem (1.7) – (1.12) we apply the method of matched asymptotic expansions [2, 3] choosing as the parameter of the expansion the small quantity β^{-1}, and we use the results given in [4 – 8].

From an analysis of Eqs. (1.7) – (1.10) for large values of β, we can show just as in [8] that the form of the asymptotic solutions differs substantially depending on the values of the parameters σ_E, σ and σ_Q, and we can separate the following particular cases:

$$^1/_2 < \sigma_Q < 1, \; (\sigma_Q + \sigma)(1 + \sigma_Q + 2\sigma)^{-1} < \sigma_E < {}^1/_2$$
$$0 < \sigma_E < (\sigma_Q + \sigma)(1 + \sigma_Q + 2\sigma)^{-1}$$

2. Solution for $^1/_2 < \sigma_E < 1$ or $1 < E_1/E_2 < \infty$. We partition the interval $0 \leqslant \tau \leqslant 1$ into two regions: a small neighborhood of $\tau = 1$ (inner region) where we introduce the variable $\tau^* = \beta(1 - \tau)$, and the remaining portion of the interval (outer region). We limit ourselves to determining two terms of the expansion of the eigenvalue μ which we seek in the form

$$\mu = (\mu_0 + \beta^{-1}\mu_1)\beta^{-2} \exp(-\beta\sigma_E) \qquad (2.1)$$

The corresponding expansions of the functions r, q, H and G in the inner and outer regions have the form

$$r(\tau^*) = r_0(\tau^*) + \beta^{-1}r_1(\tau^*) + \beta^{-2}r_2(\tau^*) \qquad (2.2)$$
$$q(\tau^*) = q_0(\tau^*) + \beta^{-1}q_1(\tau^*) + \beta^{-2}q_2(\tau^*)$$
$$H(\tau^*) = H_0(\tau^*) + \beta^{-1}H_1(\tau^*), \quad G(\tau^*) = G_0(\tau^*) + \beta^{-1}G_1(\tau^*)$$
$$r(\tau) = r_0(\tau) + \beta^{-2}r_1(\tau), \quad q(\tau) = q_0(\tau) + \beta^{-2}q_1(\tau)$$
$$H(\tau) = \overline{H}(\tau, \beta), \qquad G = \overline{G}(\tau, \beta)$$

Here, as well as in the following sections, the form of the expansions (2.1) and (2.2) is established from an analysis of the different versions and discarding of those which do not satisfy all the requirements set forth for a solution of the problem (1.1) – (1.12). The overbar is used to denote functions which, for an increase in β, decrease faster than any power of the small parameter β^{-1}, for example, according to an exponential law.

The equations for the successive terms of the expansion (2.1) and (2.2) are determined by substituting (2.1) and (2.2) into (1.7) – (1.10) and then grouping and equating terms of the same order of smallness. The outer expansions must satisfy the boundary conditions (1.11) and the inner expansions — the conditions (1.12). In addition, the outer and inner expansions must be bound by the matching condition, which is expressed by requiring equivalence of asymptotic behavior of the inner and outer expansions, represented in the form of functions of the intermediate variable [3, 5].

Substituting (2.1) and the inner expansions (2.2) into the Eqs. (1.7) – (1.10) and the boundary conditions (1.12), we obtain successively

$$\frac{dr_0}{d\tau^*}=\frac{dq_0}{d\tau^*}=0,\quad r_0(0)=q_0(0)=1,\quad r_0(\tau^*)=q_0(\tau^*)=1 \tag{2.3}$$

$$r_1(\tau^*)=q_1(\tau^*),\quad r_2(\tau^*)=q_2(\tau^*) \tag{2.4}$$
$$H_0(\tau^*)=G_0(\tau^*),\quad H_1(\tau^*)=G_1(\tau^*)$$

$$\frac{dr_1}{d\tau^*}=-L,\qquad r_1(0)=0,\qquad r_1(\tau^*)=-L\tau^* \tag{2.5}$$

$$\mu_0\frac{dH_0}{d\tau^*}=\frac{\sigma_k r_1}{1-H_0}\exp\frac{-\sigma_E\tau^*}{1+\sigma},\quad H_0(0)=1 \tag{2.6}$$

From (2.5) and (2.6) it follows that

$$H_0(\tau^*)=1-\left[\frac{2\sigma_k L(1+\sigma)^2}{\mu_0\sigma_E^2}\gamma\left(\frac{-\sigma_E\tau^*}{1+\sigma}\right)\right]^{1/2} \tag{2.7}$$

Here and in the sequel $\gamma(x)\equiv 1-(1+x)\exp(-x)$

Substituting (2.1) and the outer expansions (2.2) into (1.7) – (1.10) and (1.11), we obtain

$$\frac{dq_0}{d\tau}=\frac{Lq_0}{\tau},\quad \frac{dr_0}{d\tau}=\frac{Lr_0}{\tau},\quad r_0(0)=q_0(0)=0 \tag{2.8}$$

$$\frac{dq_1}{d\tau}=\frac{Lq_1}{\tau},\quad \frac{dr_1}{d\tau}=\frac{Lr_1}{\tau},\quad r_1(0)=q_1(0)=0 \tag{2.9}$$

Hence

$$q_0(\tau)=C_1\tau^L,\ r_0(\tau)=C_2\tau^L,\ q_1(\tau)=C_3\tau^L,\ r_1(\tau)=C_4\tau^L \tag{2.10}$$

Here and in the sequel constants of integration are denoted by C .

From the matching of the inner and outer expansions we find

$$\mu_0=2\sigma_k L(1+\sigma)^2\sigma_E^{-2},\qquad C_1=C_2=1 \tag{2.11}$$
$$H_0(\tau^*)=1-\gamma^{1/2}\left(\frac{\sigma_E\tau^*}{1+\sigma}\right)$$

For the following terms of the expansion in the inner region we obtain:

$$\frac{dr_2}{d\tau^*}=\frac{-L(r_1+\tau^*)}{1-H_0},\quad r_2(0)=0 \tag{2.12}$$
$$\frac{dH_1}{d\tau^*}=\frac{\sigma_k L\tau^*}{\mu_0(1-H_0)}\left[\frac{\mu_1}{\mu_0}-\frac{\tau^*}{1-H_0}-\frac{H_1}{1-H_0}+\frac{\sigma_E\tau^{*2}}{(1+\sigma)^2}-\frac{r_2}{r_1}\right]\exp\left(\frac{-\sigma_E\tau^*}{1+\sigma}\right),\quad H_1(0)=0$$

Hence

$$r_2(\tau^*)=\frac{(L-1)L(1+\sigma)^2}{\sigma_E^2}j_1\left(\frac{\sigma_E\tau^*}{1+\sigma}\right),\quad j_1(x)=\int_0^x\gamma^{-1/2}(t)\,tdt$$
$$H_1(\tau^*)=\frac{\sigma_k L}{\mu_0(1-H_0)}\int_0^{\tau^*}\left[\frac{\mu_1}{\mu_0}-\frac{r_2}{r_1}+\frac{\sigma_E x^2}{(1+\sigma)^2}-\frac{x}{1-H_0(x)}\right]x\exp\frac{-\sigma_E x}{1+\sigma}dx \tag{2.13}$$

Matching, we find

$$C_3=C_4=\frac{(L-1)L(1+\sigma)^2}{\sigma_E^2}j_2(\infty),\qquad j_2(x)=j_1(x)-\frac{x^2}{2} \tag{2.14}$$

$$\frac{\mu_1}{\mu_0} = \frac{2}{\sigma_E}\left[(1+\sigma)\left(\frac{j_3(\infty)}{2} + 1 - L\right) - 3\right], \quad j_3(x) = \int_0^x \gamma^{-1/2}(t)\, t^2 e^{-t} dt$$

$$j_2(\infty) = 2.92; \quad j_3(\infty) = 2.688$$

The resulting formulas give an asymptotic solution of the problem for the particular case considered. We write the two-term expression for the combustion mass rate (2.1), (2.11), (2.14) in dimensional variables

$$m = \left(\frac{2k_1 L\lambda\rho}{c}\right)^{1/2} \left(\frac{RT_+}{E_1}\right)\left(\frac{T_+}{T_+ - T_-}\right)\left\{1 + \frac{RT_+}{E_1}\left[\frac{T_+}{T_+ - T_-}(2.344 - L) - 3\right]\right\} \exp\frac{-E_1}{2RT_+} \tag{2.15}$$

The expression (2.15) establishes an analytical dependence on the combustion rate on the characteristics of the process, including the dependence on the ratio of the coefficients of thermal diffusivity and diffusion. It is evident that for the case considered the combustion rate is completely determined by the kinetic characteristics of the first reaction. Using the terminology of [9], it is natural to call this mode — the coalescence mode.

3. Solution for $(\sigma_Q + \sigma) / (1 + \sigma_Q + 2\sigma) < \sigma_E < 1/2$ or $(T_- + C^{-1}Q_1) / T_+ < E_1 / E_2 < 1$. In this case we separate on the interval $0 \leqslant \tau \leqslant 1$ two inner and two outer regions. The inner regions are small neighborhoods of the points $\tau = 1$ and $\tau = \tau_1^\circ \equiv \sigma_E(1+\sigma)(1-\sigma_E)^{-1} - \sigma$, $\sigma_Q < \tau_1^\circ < 1$. The outer regions are the segments $\tau_1^\circ < \tau < 1$ and $0 \leqslant \tau < \tau_1^\circ$. We first consider the solution in the regions $\tau \sim 1$ and $\tau_1^\circ < \tau < 1$. We seek the expansion of the eigenvalue μ, and the inner and outer expansions of the unknown functions in the form

$$\mu = (\mu_0 + \beta^{-1}\mu_1)\beta^{-2} \exp[-\beta(1-\sigma_E)] \tag{3.1}$$

$$\begin{aligned}
&q(\tau^*) = q_0(\tau^*) + \beta^{-1}q_1(\tau^*) + \beta^{-2}q_2(\tau^*) \\
&G(\tau^*) = G_0(\tau^*) + \beta^{-1}G_1(\tau^*) \\
&r(\tau^*) = 1 + \bar{r}(\tau^*, \beta), \ H(\tau^*) = 1 + \bar{H}(\tau^*, \beta), \ \tau^* = \beta(1-\tau) \qquad (3.2) \\
&q(\tau) = q'_0(\tau) + \beta^{-2}q'_1(\tau), \qquad G(\tau) = \bar{G}(\tau, \beta) \\
&r(\tau) = 1 + \bar{r}(\tau, \beta), \qquad H(\tau) = 1 + \bar{H}(\tau, \beta)
\end{aligned}$$

Substituting (3.1) and (3.2) into (1.7) – (1.10) and separating terms of the same order of smallness, and taking the boundary coditions into account, we can obtain in the inner region

$$\frac{dq_0}{d\tau^*} = 0, \qquad q_0(0) = 1, \qquad q_0(\tau^*) = 1 \tag{3.3}$$

$$\frac{dq_1}{d\tau^*} = -L(1-\sigma_Q)^{-1}, \qquad q_1(0) = 0, \ q_1(\tau^*) = -L(1-\sigma_Q)^{-1}\tau^* \tag{3.4}$$

$$\frac{dq_2}{d\tau^*} = \frac{-L[(1-\sigma_Q)q_1 + \tau^*]}{(1-\sigma_Q)^2(1-G_0)}, \qquad q_2(0) = 0 \tag{3.5}$$

$$\mu_0\frac{dG_0}{d\tau^*} = \frac{(1-\sigma_k)q_1}{(1-\sigma_Q)(1-G_0)}\exp\frac{-(1-\sigma_E)\tau^*}{1+\sigma}, \qquad G_0(0) = 1 \tag{3.6}$$

$$\begin{aligned}
\frac{dG_1}{d\tau^*} = {} & \frac{(1-\sigma_k)L\tau^*}{\mu_0(1-\sigma_Q)^2(1-G_0)}\left[\frac{\mu_1}{\mu_0} - \frac{\tau^*}{(1-\sigma_Q)(1-G_0)} - \frac{G_1}{1-G_0} + \right. \\
& \left.\frac{(1-\sigma_E)\tau^{*2}}{(1+\sigma)^2} - \frac{q_2}{q_1}\right]\exp\frac{-(1-\sigma_E)\tau^*}{1+\sigma}, \qquad G_1(0) = 0
\end{aligned} \tag{3.7}$$

and in the outer region

$$\frac{dq_0'}{d\tau} = \frac{Lq_0'}{\tau - \sigma_Q}, \quad \frac{dq_1'}{d\tau} = \frac{Lq_1'}{\tau - \sigma_Q} \tag{3.8}$$
$$q_0' = C_5 (\tau - \sigma_Q)^L, \quad q_1' = C_6 (\tau - \sigma_Q)^L$$

From (3.4) and (3.6) we find

$$G_0(\tau^*) = 1 - \left\{\frac{2L(1-\sigma_k)(1+\sigma)^2}{\mu_0(1-\sigma_Q)^2(1-\sigma_E)^2}\gamma\left(\frac{-(1-\sigma_E)\tau^*}{1+\sigma}\right)\right\}^{1/2} \tag{3.9}$$

After matching of inner and outer expansions we obtain

$$\mu_0 = \frac{2L(1-\sigma_k)(1+\sigma)^2}{(1-\sigma_Q)^2(1-\sigma_E)^2} \tag{3.10}$$
$$C_5 = \frac{1}{(1-\sigma_Q)^L}, \quad G_0(\tau^*) = 1 - \gamma^{1/2}\left(\frac{(1-\sigma_E)\tau^*}{1+\sigma}\right)$$

Next, taking note of (3.4) and (3.9), we find from (3.5)

$$q_2(\tau^*) = \frac{(L-1)L(1+\sigma)^2}{(1-\sigma_E)^2(1-\sigma_Q)^2} j_1\left[\frac{(1-\sigma_E)\tau^*}{1+\sigma}\right] \tag{3.11}$$

Here the function j_1 is defined in (2.13). Integrating (3.7), we obtain

$$G_1(\tau^*) = \frac{(1-\sigma_k)L}{\mu_0(1-\sigma_Q)^2(1-G_0)}\int_0^{\tau^*}\left[\frac{\mu_1}{\mu_0} - \frac{q_2}{q_1} + \frac{(1-\sigma_E)x^2}{(1+\sigma)^2} - \right. \tag{3.12}$$
$$\left.\frac{x}{(1-\sigma_Q)(1-G_0)}\right] x \exp\frac{-(1-\sigma_E)x}{1+\sigma}\,dx$$

As a result of matching we have

$$\frac{\mu_1}{\mu_0} = \frac{2}{1-\sigma_E}\left[\frac{1+\sigma}{1-\sigma_Q}\left(\frac{j_3(\infty)}{2} + 1 - L\right) - 3\right] \tag{3.13}$$
$$C_6 = \frac{(L-1)L(1+\sigma)^2}{(1-\sigma_E)^2(1-\sigma_Q)^{2+L}} j_2(\infty)$$

Here the function j_3 is defined in (2.14). Using (3.1), (3.10) and (3.13), we write the asymptotic two-term expression for the combustion rate in dimensional variables

$$m = \left(\frac{2K_2L\lambda\rho}{c}\right)^{1/2}\left(\frac{RT_+}{E_2}\right)\left(\frac{cT_+}{Q_2}\right)\left\{1 + \left(\frac{RT_+}{E_2}\right) \times \right. \tag{3.14}$$
$$\left.\left[\frac{T_+}{Q_2}(2.344 - L) - 3\right]\right\} \exp\frac{-E_2}{2RT_+}$$

In order to complete the construction of the solution it is necessary to determine the functions r, q, H and G in the regions $0 < \tau < \tau_1^\circ$ and $\tau \sim \tau_1^\circ$.

Variation of the functions r and H from zero to unity occurs mainly in a narrow zone close to $\tau = \tau_1^\circ = (1-\sigma_E)^{-1}\sigma_E(1+\sigma) - \sigma$ or $T = T_1^\circ = T_+E_1E_2^{-1}$, where both sides of Eq. (1.9) become equal in order of magnitude. We seek solutions close to $\tau = \tau_1^\circ$ in the form of inner expansions

$$r(\tau_1^*) = r_0(\tau_1^*) + \beta^{-1}r_1(\tau_1^*) + \beta^{-2}r_2(\tau_1^*) \tag{3.15}$$
$$H(\tau_1^*) = H_0(\tau_1^*) + \beta^{-1}H_1(\tau_1^*)$$

$$q(\tau_1^*) = q_0(\tau_1^*) + \beta^{-1}q_1(\tau_1^*) + \beta^2 r_2(\tau_1^*)$$
$$G(\tau_1^*) = \overline{G(\tau_1^*, \beta)}, \qquad \tau_1^* = \beta(\tau_1^\circ - \tau)$$

Since point $\tau = \tau_1^\circ$ lies within the interval $[0,1]$, hence the expansions (3.15) must satisfy the matching conditions with the corresponding expansions in the two outer regions. In the outer region $\tau_1^\circ > 0$ the solution is determined by the formulas (3.2), (3.8), (3.10), (3.13), and

$$q(\tau) = \left(\frac{\tau - \sigma_Q}{1 - \sigma_Q}\right)^L + \beta^{-2}\frac{(1+\sigma)^2 L(L-1)}{(1-\sigma_E)^2(1-\sigma_Q)^{2+L}} j_2(\infty)(\tau - \sigma_Q)^L \tag{3.16}$$

$$G_1(\tau) = \overline{G(\tau, \beta)}, \; H(\tau) = 1 + \overline{H(\tau, \beta)}, \; r(\tau) = 1 + r(\tau, \beta)$$

In the outer region $\tau < \tau_1^\circ$ just as in Sect. 2,

$$r(\tau) = r_0(\tau) + \beta^{-2}r_1(\tau), \; q(\tau) = q_0(\tau) + \beta^{-2}q_1(\tau) \tag{3.17}$$
$$r_0(\tau) = C_7\tau^L, \; r_1(\tau) = C_8\tau^L, \; q_0(r) = C_9\tau^L, \; q_1(\tau) = C_{10}\tau^L$$
$$H(\tau) = \overline{H(\tau, \beta)}, \qquad G(\tau) = \overline{G(\tau, \beta)}$$

Substituting (3.15) into the system (1.7) – (1.10) for $r_0(\tau_1^*)$ and $q_0(\tau_1^*)$, we obtain the equations

$$dr_0 / d\tau_1^* = dq_0 / d\tau_1^* = 0 \tag{3.18}$$

The solution of Eq. (3.18), satisfying the matching condition with (3.16), has the form

$$r_0(\tau_1^*) = 1, \qquad q_0(\tau_1^*) = (1 - \sigma_Q)^{-L}(\tau_1^\circ - \sigma_Q)^L \tag{3.19}$$

Matching of (3.19) with (3.17) yields

$$C_7 = (\tau_1^\circ)^{-L}, \qquad C_9 = (\tau_1^\circ)^{-L}(1 - \sigma_Q)^{-L}(\tau_1^\circ - \sigma_Q)^L$$

Further, for $r_1(\tau_1^*)$, $q_1(\tau_1^*)$ and $H_0(\tau_1^*)$ we can obtain the equations

$$\frac{dr_1}{d\tau_1^*} = \frac{-L(1 - H_0)}{\tau_1^\circ - \sigma_Q H_0}, \quad \frac{dq_1}{d\tau_1^*} = \frac{-L(\tau_0 - \sigma_Q)^L}{(1 - \sigma_Q)^L(\tau_1^\circ - \sigma_Q H_0)} \tag{3.20}$$

$$\mu_0\frac{dH_0}{d\tau_1^*} = \frac{\sigma_k r_1}{\tau_1^\circ - \sigma_Q H_0}\exp\frac{-\sigma_E(1+\sigma)\tau_1^*}{(\tau_1^\circ + \sigma)^2}$$

Analyzing (3.20), we should take into account the fact that the value of the parameter μ_0 has already been established earlier and is determined by the formula (3.10). We can prove that subject to the condition $\tau_1^\circ > \sigma_Q$ the system of equations (3.20) has a solution (unique) which satisfies the matching conditions with (3.16) and (3.17). Determination of this solution is only possible by numerical integration.

We can write a system of equations for $r_1(\tau_1^*)$, $q_1(\tau_1^*)$ and $H_1(\tau_1^*)$ analogous to (3.20). For determining the eigenvalue of the problem which is the principal aim of the study, there is no need in the solution of this system, as is also the case for the solution of the system (3.20).

As the formula (3.14) shows, the combustion rate for the case considered is determined by the kinetic characteristics of the second reaction. The zones of the two consecutive stages of the chemical conversion are separated by a spatial and a temperature interval and are related by a heat flux.

4. Solution for $0 \leqslant \sigma_E < (\sigma + \sigma_Q)(1 + \sigma_Q + 2\sigma)^{-1}$ or $0 < E_1 E_2^{-1} < (T_- + c^{-1}Q_1)T_+^{-1}$. In this case the functions $H(\tau)$ and $r(\tau)$, just as in Sect. 3, vary mainly in a narrow zone close to $\tau = \tau_1^\circ$, outside of which, to within exponential terms, they are equal to zero and unity. However now the position of the point τ_1° is independent of σ_E and is determined from the equation $\tau_1^\circ = \sigma_Q$. The behavior of the functions $G(\tau)$ and $q(\tau)$ differs substantially from that in Sect. 3. The eigenvalue μ is to be sought in the form

$$\mu = (\mu_0 + \beta^{-1}\mu_1)\beta^{-2}\exp\frac{-\beta\sigma_E(1+\sigma)}{\sigma_Q + \sigma} \tag{4.1}$$

In constructing the solution it is sufficient to consider three regions of distinct behavior for r, q, H and G. The outer region $0 \leqslant \tau < \sigma_Q$, the inner region consisting of a small neighborhood of the point $\tau = \tau_1^\circ = \sigma_Q$, and the outer region $\sigma_Q < \tau \leqslant 1$.

In the outer region $0 \leqslant \tau < \sigma_Q$

$$r(\tau) = r_0(\tau) + \beta^{-2}r_1(\tau),\quad q(\tau) = \overline{q(\tau,\beta)} \qquad G(\tau) = \overline{G(\tau,\beta)},\quad H(\tau) = \overline{H(\tau,\beta)} \tag{4.2}$$

Substituting (4.2) into (1.7) – (1.10), we can find

$$r(\tau) = C_{11}\tau^L + \beta^{-2}C_{12}\tau^L \tag{4.3}$$

In the inner region $\tau \sim \sigma_Q$ we introduce the variable $\tau_1^* = \beta(\sigma_Q - \tau)$ and we consider separately solutions for $\tau_1^* > 0$ $(\tau < \sigma_Q)$ and $\tau_1^* < 0$ $(\tau > \sigma_Q)$

$$\tau_1^* > 0,\quad r(\tau_1^*) = r_0^-(\tau_1^*) + \beta^{-1}r_1^-(\tau_1^*) + \beta^{-2}r_2^-(\tau_1^*) \tag{4.4}$$
$$H_0(\tau_1^*) = H_0^-(\tau_1^*) + \beta^{-1}H_1^-(\tau_1^*),\quad q(\tau_1^*) = q^-(\tau_1^*,\beta)$$
$$G(\tau_1^*) = \overline{G^-(\tau_1^*,\beta)}$$
$$\tau_1^* < 0,\quad r(\tau_1^*) = 1 + \overline{r^+(\tau_1^*,\beta)},\quad H(\tau_1^*) = 1 + \overline{H^+(\tau_1^*,\beta)} \tag{4.5}$$
$$q(\tau_1^*) = \beta^{-1}q_1^+(\tau_1^*) + q_2^+(\tau_1^*,\beta),\quad G(\tau_1^*) = \beta^{-1}G_1^+(\tau_1^*) + G_2^+(\tau_1^*,\beta)$$

In constructing the solution in the inner region we take into account that the point $\tau = \sigma_Q$, $H = r = 1$, $G = q = 0$ is a singular point and we use the conditions [8]

$$\tau_1^* = 0,\quad H_0^-(0) = r_0^-(0) = 1 \qquad r_1^-(0) = r_2^-(0) = \overline{H}_1(0) = q_1^+(0) = G_1^+(0) = 0 \tag{4.6}$$

In the outer region $\sigma_Q < \tau < 1$

$$r(\tau) = 1 + \overline{r'(\tau,\beta)},\quad H(\tau) = 1 + \overline{H'(\tau,\beta)} \qquad q(\tau) = \frac{\tau - \sigma_Q}{1 - \sigma_Q} + \overline{q'(\tau,\beta)},\quad G(\tau) = \frac{\tau - \sigma_Q}{1 - \sigma_Q} + \overline{G'(\tau,\beta)} \tag{4.7}$$

Substituting (4.1), (4.4) into (1.7) – (1.10), we find

$$\frac{dr_0^-}{d\tau_1^*} = 0,\ \frac{dr_1^-}{d\tau_1^*} = \frac{-L}{\sigma_Q},\quad \mu_0\frac{dH_0^-}{d\tau_1^*} = \frac{\sigma_K r_1^-}{\sigma_Q(1 - H_0^-)}\exp\frac{-\sigma_E(1+\sigma)}{(\sigma_Q+\sigma)^2}\tau_1^* \tag{4.8}$$

From (4.8), taking into account the matchimg conditions and (4.6), we have

$$r_0^-(\tau_1^*) = 1,\qquad r_1^-(\tau_1^*) = \frac{-L\tau_1^*}{\sigma_Q},\qquad \mu_0 = \frac{2\sigma_K L(\sigma_Q+\sigma)^4}{\sigma_Q^2\sigma_E^2(1+\sigma)^2} \tag{4.9}$$

$$C_{11} = \sigma_Q^{-L}, \qquad H_0^-(\tau_1^*) = 1 - \gamma^{1/2}\left(\frac{\sigma_E(1+\sigma)\tau_1^*}{(\sigma_Q+\sigma)^2}\right)$$

For the following terms of the expansions for $\tau_1^* > 0$ we can obtain

$$\frac{dr_2^-}{d\tau_1^*} = \frac{-L(\sigma_Q r_1^- + \tau_1^*)}{\sigma_Q^2(1-H_0^-)}, \quad \frac{dH_1^-}{d\tau_1^*} = \frac{\sigma_k L\tau_1^*}{\sigma_Q^2(1-H_0)}\left[\frac{\mu_1}{\mu_0} - \frac{r_2^-}{r_1^-} - \frac{\tau_1^*}{\sigma_Q(1-H_0^-)} - \frac{H_1^-}{1-H_0^-} + \frac{\sigma_E(1+\sigma)}{(\sigma_Q+\sigma)^3}\tau_1^{*2}\right]\exp\frac{-\sigma_E(1+\sigma)\tau_1^*}{(\sigma_Q+\sigma)^2} \tag{4.10}$$

From (4.10), taking note of (4.6) and the matching conditions we find

$$r_2^-(\tau_1^*) = \frac{L(L-1)(\sigma_Q+\sigma)^4}{\sigma_Q^2(1+\sigma)^2\sigma_E^2}\, j_1\left(\frac{\sigma_E(1+\sigma)\tau_1^*}{(\sigma_Q+\sigma)^2}\right)$$

$$\frac{\mu_1}{\mu_0} = \frac{2(\sigma_Q+\sigma)^2}{\sigma_E(1+\sigma)}\left[\frac{\sigma_Q+\sigma}{\sigma_Q}\left(\frac{j_3(\infty)}{2} + 1 - L\right) - 3\right]$$

$$H_1^-(\tau_1^*) = \frac{\sigma_K L}{\mu_0\sigma_Q^2(1-H_0^-)}\int_0^{\tau^*}\left[\frac{\mu_1}{\mu_0} - \frac{r_2^-}{r_1} + \frac{\sigma_E(1+\sigma)}{(\sigma_Q+\sigma)^3}x^2 - \frac{x}{\sigma_Q(1-H_0)}\right]x\exp\frac{-\sigma_E(1+\sigma)x}{(\sigma_Q+\sigma)^2}\,dx$$

We write the two-term formula for the combustion rate in dimensional variables

$$m = \left(\frac{2k_1L\lambda\rho}{c}\right)^{1/2}\left(\frac{RT_+^{(1)}}{E_1}\right)\left(\frac{T_+^{(1)}}{T_+^{(1)} - T_-}\right)\left\{1 + \frac{RT_+^{(1)}}{E_1}\times\left[\frac{T_+^{(1)}}{T_+^{(1)} - T_-}(2.344 - L) - 3\right]\right\}\exp\frac{-E_1}{2RT_+^{(1)}}, \quad T_+^{(1)} = T_- + \frac{Q_1}{C} \tag{4.11}$$

Substituting (4.1), (4.5) into (1.7) – (1.10), we can obtain

$$q_1^+(\tau_1^*) = G_1^+(\tau_1^*) = -(1-\sigma_Q)^{-1}\tau_1^* \tag{4.12}$$

The functions (4.12) satisfy the matching condition with (4.6).

The formulas we have obtained above give a complete solution of the problem for the special case considered. From (4.11) it follows that the combustion rate is determined here by the characteristics of the first reaction, the front of which propagates indedendently of the second reaction which takes place under induction conditions. Using the terminology of [9], we call this combustion mode a mode of separation.

5. Discussion of the results. The analysis presented here has enabled us to identify the characteristic combustion modes and also the regions over which the parameters of the problem vary. The approximate analytical expressions obtained for the combustion rate and the distribution of the parameters completely define the dependence of the combustion rate and the wave structure on the physico-chemical characteristics of the burning mixture. In particular, the results include the special case $L = 1$, for which we have the additional integral $\tau = r\sigma_Q + (1-\sigma_Q)q$.

A comparison of our data with the numerical results obtained in [10, 11] for $L = 1$

shows good agreement. The characteristic combustion modes identified through an asymptotic examination, which were obtained numerically in [10], show that the regions of their occurrence coincide, with acceptable accuracy, with those found in [10]. The applicability of our results is not restricted to the case of ultimate large β. Similarly to [4 – 8], the data obtained describe the process with sufficient accuracy, also for values of β substantially less than ten.

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